Resistor networks

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0.1 What this is

This short note summarizes my thoughts on the linear algebraic formulation of resistor network analysis with respect to internal currents, potentials and total resistance. The underlying physical principles are Ohm's and Kirchhoff's laws and the whole idea is nothing new. A slightly alternative (but equivalent) formulation can be found in [1, pp. 426].

0.2 Problem statement

Consider a graph of n nodes, possibly connected by single electric resistors satisfying Ohm's law. Denote by $C_{ij} > 0$ the conductance (i.e. inverse resistance) of the resistor directly connecting nodes $i \neq j$ (equal to zero if the two nodes are not connected). Note that $C_{ij} = C_{ji}$. Let U_i be the potential at node i and $I_{ij} = (U_i - U_j) \cdot C_{ij}$ be the current flowing from node i to node j via their directly connecting resistor. For each node, let either an externally fixed potential $U_i \stackrel{!}{=} U_i^*$ be applied (e.g. grounded) or the net charge outflux $\sum_{j \neq i} I_{ij} =: I_i \stackrel{!}{=} I_i^*$ be given (e.g. positive if *output* node, negative if *input* node and by Kirchhoff, zero otherwise). W.l.o.g. we shall order nodes in that way that, the first k nodes have given potentials U_i^*, and the remaining $(n-k)$ nodes have given net outflux I_i^*. Note that by Kirchhoff, one needs to impose the consistency condition

$$\sum_{i=k+1}^{n} I_k^* = 0. \tag{0.1}$$

We wish to solve find all potentials U_i and currents I_{ij} within the network, subject to the above conditions. To find the effective resistance $\overline{R}_{12}$ between any two nodes, say, node 1 and 2, one would have to solve the problem with the conditions $U_1^* = V$, $U_2^* = 0$, $I_i^* = 0$ $\forall i \geq 3$ and set $\overline{R}_{12} := V/I_1$ (note that by (0.1) one has $I_2 = -I_1$, i.e. what flows out of 1 flows into 2).

0.3 Formulation as linear algebraic problem

The above conditions read

$$\begin{aligned} &U_i = U_i^* \quad \forall i \leq k, \\ &\sum_{j \neq i} (U_i - U_j) \cdot C_{ij} = I_i^* \quad \forall i > k \end{aligned} \tag{0.2}$$

and represent a linear system of equations in the unknown potentials U_i. In matrix form,

$$\boxed{\mathbb{A} \cdot \mathbf{U} = \mathbf{b},} \tag{0.3}$$

where

$$A_{ij} = \begin{cases} \delta_{ij} & : i \le k \\ C_i \delta_{ij} - C_{ij} \cdot \overline{\delta}_{ij} & : i > k \end{cases}, \quad C_i := \sum_{j \ne i} C_{ij}, \quad \mathbf{b} := \begin{pmatrix} U_1^* \\ \vdots \\ U_k^* \\ I_{k+1}^* \\ \vdots \\ I_n^* \end{pmatrix}. \tag{0.4}$$

Note that we designate $\overline{\delta}_{ij} := (1 - \delta_{ij})$, where δ_{ij} is the Kronecker symbol. The currents I_{ij} between nodes $i \ne j$ via their connecting resistors are given by

$$I_{ij} = (U_i - U_j) \cdot C_{ij}, \quad i \ne j, \tag{0.5}$$

or in matrix form

$$\boxed{\mathbb{I} = \mathrm{diag}(\mathbf{U}) \cdot \mathbb{C} - \mathbb{C} \cdot \mathrm{diag}(\mathbf{U}),} \tag{0.6}$$

where

$$\mathbb{I} := \begin{pmatrix} I_{11} & \dots & I_{1n} \\ \vdots & \ddots & \vdots \\ I_{n1} & \dots & I_{nn} \end{pmatrix}, \quad \mathbb{C} := \begin{pmatrix} C_{11} & \dots & C_{1n} \\ \vdots & \ddots & \vdots \\ C_{n1} & \dots & C_{nn} \end{pmatrix}, \quad I_{ii} := 0, \quad C_{ii} := 0. \tag{0.7}$$

0.4 Example 01: Given potential gradient

Consider a network of 4 nodes, with node 1 being subject to the externally imposed potential $U_1^* = V$ and node 2 being grounded, i.e. $U_2^* = 0$. All other nodes are neither sinks nor sources, so that $I_3^* = 0 = I_4^*$. Suppose every node i is connected to every node $j \ne i$ with a resistor of equal conductance $C_{ij} = C$. See Fig. 0.1 for an illustration.

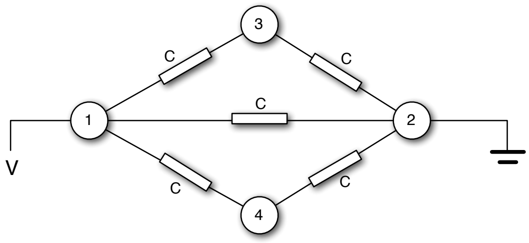

Figure 0.1: On example 01.

Equation (0.3) reads

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -C & -C & 3C & -C \\ -C & -C & -C & 3C \end{pmatrix}}_{=\mathbb{A}} \cdot \begin{pmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{pmatrix} = \begin{pmatrix} V \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{0.8}$$

Its unique solution is

$$\begin{pmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{pmatrix} = \begin{pmatrix} V \\ 0 \\ V/2 \\ V/2 \end{pmatrix}. \tag{0.9}$$

The currents I_{ij} through the resistors are given by (0.6). In particular

$$I_{12} = VC, \quad I_{13} = VC/2, \quad I_{14} = VC/2, \tag{0.10}$$

so that $I_1 = 2VC$. Hence, the effective resistance between node 1 and 2 is given by $\overline{R}_{12} = V/I_1 = 1/(2C)$.

0.5 Example 02: Given net current throughput

Now consider the same network of 4 nodes, where every node is connected to every node via a resistor of equal conductance C. Let node 1 be a *source* and node 2 a *sink*, i.e. with a current $I_1^* = I$ fed into the network through 1 and extracted through node 2, all other nodes being neither sinks nor sources. Note that by (0.1) $I_2^* = -I_1^*$. Then (0.3) reads

$$\underbrace{C \cdot \begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{pmatrix}}_{=\mathbb{A}} \cdot \begin{pmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{pmatrix} = \begin{pmatrix} I \\ -I \\ 0 \\ 0 \end{pmatrix}. \tag{0.11}$$

Its solutions are

$$\begin{pmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{pmatrix} = \begin{pmatrix} U \\ U \\ U \\ U \end{pmatrix} + \begin{pmatrix} I/(4C) \\ -I/(4C) \\ 0 \\ 0 \end{pmatrix}, \tag{0.12}$$

where U is some arbitrary base potential. The currents I_{ij} through the resistors are given by (0.6). In particular

$$I_{12} = I/2, \quad I_{13} = I/4, \quad I_{14} = I/4, \tag{0.13}$$

so that indeed $I_1 = I$. Hence, the effective resistance between node 1 and 2 is given by $\overline{R}_{12} = (U_1 - U_2)/I = 1/(2C)$, in accordance with the previous example.

References

[1] G. Strang (2009), *Introduction to Linear Algebra, 4th Ed.*
Wellesley Cambridge Press